DERIVATION OF THE NAVIER STOKES EQUATION

1. CAUCHY'S EQUATION

First we derive **Cauchy's equation** using Newton's second law.

We take a differential fluid element. We consider the element as a *material element* (instead of a control volume) and apply Newton's second law

$$m \cdot \vec{a} = \sum \vec{F}$$

or since $\vec{a}(t) = \frac{D\ \overrightarrow{V(t)}}{Dt}$

$$m \cdot \frac{D\ \vec{V}}{Dt} = \sum \vec{F} \qquad (Eq\ 1)$$

We express the total force as the sum of body forces and surface forces

$\sum \vec{F} = \sum \vec{F}_{body} + \sum \vec{F}_{surface}$. Thus $(Eq\ 1)$ can be written as

$$m \cdot \frac{D\ \vec{V}}{Dt} = \sum \vec{F}_{body} + \sum \vec{F}_{surface} \qquad (Eq\ 2)$$

Body forces:	**Surface forces:**
Gravity force	Pressure forces
Electromagnetic forse	Viscous forces
Centrifugal force	
Coriolis force	

We cosider the x-component of $(Eq\ 2)$.

Since $m = \rho dxdydz$ and $\vec{V} = (u, v, w)$ we have

$$\rho dxdydz \cdot \frac{Du}{Dt} = \sum \vec{F}_{x,body} + \sum \vec{F}_{x,surface} \qquad (Eq\ 3)$$

We denote the stress tensor σ_{ij} (pressure forces+ viscous forces)

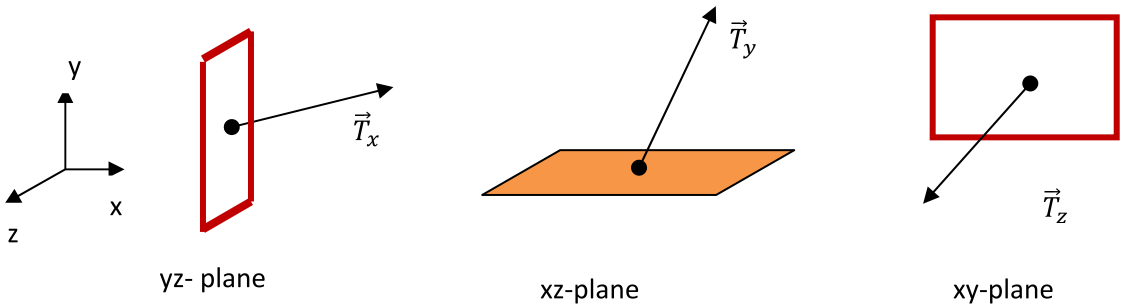
$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \quad ,$$

the viscous stress tensor $\tau_{ij} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$

and strain (deformation) rate tensor ε_{ij} where

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z} \end{bmatrix}$$

Let $\vec{T}_x = (\sigma_{xx}, \sigma_{xy}, \sigma_{xz})$, $\vec{T}_y = (\sigma_{yx}, \sigma_{yy}, \sigma_{yz})$, $\vec{T}_z = (\sigma_{zx}, \sigma_{zy}, \sigma_{zz})$ be stress vectors on the planes perpendicular to the coordinate axes.

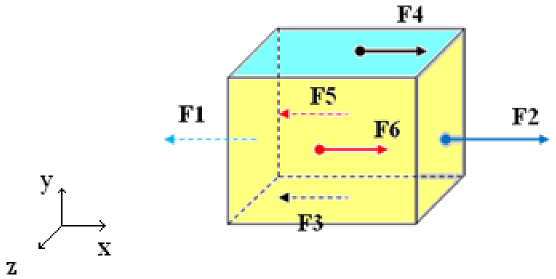


Than the stress *vector* $\vec{F}$ at any point associated with a plane of normal vector $\vec{n} = (n_1, n_2, n_3)$

can be expressed as

$$\vec{F} = n_1\vec{T}_x + n_2\vec{T}_y + n_3\vec{T}_z = (n_1, n_2, n_3) \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}.$$

We consider the net surface force in the x-direction $\sum \vec{F}_{x,surface}$ using the figure below.



Using Taylor's formula we get

$$\mathbf{F}1 = -(\sigma_{xx} - \frac{dx}{2}\frac{\partial \sigma_{xx}}{\partial x})dydz \qquad \mathbf{F}2 = (\sigma_{xx} + \frac{dx}{2}\frac{\partial \sigma_{xx}}{\partial x})dydz$$

$$\mathbf{F}3 = -(\sigma_{yx} - \frac{dy}{2}\frac{\partial \sigma_{yx}}{\partial y})dxdz \qquad \mathbf{F}4 = (\sigma_{yx} + \frac{dy}{2}\frac{\partial \sigma_{yx}}{\partial y})dxdz$$

$$\mathbf{F}5 = -(\sigma_{zx} - \frac{dz}{2}\frac{\partial \sigma_{zx}}{\partial z})dxdy \qquad \mathbf{F}6 = (\sigma_{zx} + \frac{dz}{2}\frac{\partial \sigma_{zx}}{\partial z})dxdy$$

Thus

$$\sum \vec{F}_{x,surface} = \mathbf{F}1 + \mathbf{F}2 + \mathbf{F}3 + \mathbf{F}4 + \mathbf{F}5 + \mathbf{F}6 = (\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z})dxdydz$$

If we assume that the only body force is the gravity force, we have

$$\sum \vec{F}_{x,body} = m \cdot g_x = \rho \cdot dxdydz \cdot g_x$$

Now from $(Eq\ 3)$

$$\rho dxdydz \cdot \frac{Du}{Dt} = \sum \vec{F}_{x,body} + \sum \vec{F}_{x,surface} \qquad (Eq\ 3)$$

we have

$$\rho \cdot dxdydz \cdot \frac{Du}{Dt} = \rho \cdot dxdydz \cdot g_x + (\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z})dxdydz$$

We divide by $dxdydz$ and get the equation for the *x-component:*

$$\rho \cdot \frac{Du}{Dt} = \rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}$$

or

$$\rho \cdot (\frac{\partial u}{\partial t} + u\ \frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}) = \rho g_x\ + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} \qquad \textbf{eq x}$$

In the similar way we derive the following equations for

y component:

$$\rho \cdot (\frac{\partial v}{\partial t} +\ u\frac{\partial v}{\partial x} + w\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}) = \rho g_y\ + \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} \qquad \textbf{eq y}$$

z component:

$$\rho \cdot (\frac{\partial w}{\partial t} +\ u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}) = \rho g_z\ + \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \qquad \textbf{eq z}$$

Equations **eq x,y,z,** are called **Cauchy's equations**.

THE NAVIER STOKES EQUATION

When considering $\sum \vec{F}_{x,surface}$ we can separate x components of pressure forces

and viscous forces:

$$\frac{\partial \sigma_{xx}}{\partial x} = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x}, \quad \frac{\partial \sigma_{yx}}{\partial y} = \frac{\partial \tau_{yx}}{\partial y}, \quad \frac{\partial \sigma_{zx}}{\partial z} = \frac{\partial \tau_{zx}}{\partial z}$$

In the similar way we can change y-component and z-component

Thus **Cauchy's equations** become

$$\rho \cdot (\frac{\partial u}{\partial t} + u\ \frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}) = \rho g_x\ - \frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \qquad \textbf{eq A}$$

In the similar way we derive the following equations for

y component:

$$\rho \cdot (\frac{\partial v}{\partial t} +\ u\frac{\partial v}{\partial x} + w\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}) = \rho g_y\ - \frac{\partial P}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \qquad \textbf{eq B}$$

z component:

$$\rho \cdot (\frac{\partial w}{\partial t} +\ u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}) = \rho g_z\ - \frac{\partial P}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \qquad \textbf{eq C}$$

According o the NEWTON'S LOW OF VISCOSITY the viscous stress components are related (throw a linear combination) to the (first) dynamic viscosity μ and the second viscosity λ.

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} +\ \lambda div\vec{V}\ , \qquad \tau_{xy} = \mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})\ , \qquad \tau_{xz} = \mu(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) \qquad (*)$$

$\tau_{yx} = \mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$, $\tau_{yy} = 2\mu\frac{\partial v}{\partial y} + \lambda div\vec{V}$, $\tau_{yz} = \mu(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})$ (**)

$\tau_{zx} = \mu(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})$, $\tau_{zx} = \mu(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})$ $\tau_{zz} = 2\mu\frac{\partial w}{\partial y} + \lambda div\vec{V}$ (***)

We substitute this values τ_{ij} in to **Cauchy's equations eq A, B, C** and get

THE NAVIER STOKES EQUATIONS for the compressible flow:

x-component:

$$\rho \cdot (\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}) = \rho g_x - \frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x} + \lambda div\vec{V}\right] + \frac{\partial}{\partial y}\left[\mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})\right] + \frac{\partial}{\partial z}\left[\mu(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})\right]$$

y-component:

$$\rho \cdot (\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + w\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}) = \rho g_y - \frac{\partial P}{\partial y} + \frac{\partial}{\partial x}\left[\mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})\right] + \frac{\partial}{\partial y}\left[2\mu\frac{\partial v}{\partial y} + \lambda div\vec{V}\right] + \frac{\partial}{\partial z}\left[\mu(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})\right]$$

z-component:

$$\rho \cdot (\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}) = \rho g_z - \frac{\partial P}{\partial z} + \frac{\partial}{\partial x}\left[\mu(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})\right] + \frac{\partial}{\partial y}\left[\mu(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})\right] + \frac{\partial}{\partial z}2\mu\frac{\partial w}{\partial y} + \lambda div\vec{V}$$

Remark: For a compressible flow we have $div\vec{V} = 0$ and hence from (*), (**) and (***)

$$\tau_{ij} = 2\mu\varepsilon_{ij}$$

where ε_{ij} is the strain rate tensor for the velocity field $\vec{V} = (u, v, w)$ in Cartesian coordinates:

$$\tau_{ij} = 2\mu\varepsilon_{ij} = 2\mu \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z} \end{bmatrix}$$

$$= \begin{bmatrix} 2\mu\frac{\partial u}{\partial x} & \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & 2\mu\frac{\partial v}{\partial y} & \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & 2\mu\frac{\partial w}{\partial z} \end{bmatrix}.$$

In the case when we consider an incompressible , isothermal Newtonian flow (**density** ρ **=const, viscosity** μ **=const**), with a velocity field $\vec{V} = (u(x,y,z),\ v(x,y,z),\ w\ (x,y,z))$

we can simplify the **Navier-Stokes equations** to his form:

x component:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

y- *component:*

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial P}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

z component:

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

[The vector form for these equations: $\rho\frac{D\vec{V}}{Dt} = -\nabla P + \rho\vec{g} + \mu\nabla^2\vec{V}$]