Exercises 3

M/M/1 Queueing System is a single-server queueing system with Poisson input, exponential service times and unlimited number of waiting positions.

Thus, an M/M/1 system has the following characteristics:

1. There is a single server with exponential service times and the service rate $\mu$ customers per time unit
2. Customers arriving according a Poisson process with the arrival rate $\lambda$ customers per time unit
3. Number of waiting positions = $\infty$

Figure 1. Rate transition diagram for an M/M/1 Queueing System,

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>Arrival rate = $\lambda$</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Service rate = $\mu$</td>
</tr>
<tr>
<td>$\infty$</td>
<td>Unlimited number of waiting positions</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
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</tr>
<tr>
<td>$N$</td>
<td>Average number of customers in the system, $N = N_q + N_s$</td>
</tr>
<tr>
<td>$N_q$</td>
<td>Average number of customers in the queue</td>
</tr>
<tr>
<td>$N_s$</td>
<td>Average number of customers in the service facilities</td>
</tr>
<tr>
<td>$\bar{x}$</td>
<td>Random variable which describes time spent in the service facility by a customer</td>
</tr>
<tr>
<td>$\bar{x}$</td>
<td>Average service time for a customer, $\bar{x} = E(\bar{x})$</td>
</tr>
<tr>
<td>$\bar{w}$</td>
<td>Random variable which describes time spent in the waiting queue by a customer</td>
</tr>
<tr>
<td>$W$</td>
<td>Average waiting time spent in the queue by a customer, $W = E(\bar{w})$</td>
</tr>
<tr>
<td>$\bar{s}$</td>
<td>Random variable which describes time spent in the system by a customer; $\bar{s} = \bar{x} + \bar{w}$</td>
</tr>
<tr>
<td>$T$</td>
<td>Average time spent in the system by a customer, $T = E(\bar{s})$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Arrival rate</td>
</tr>
<tr>
<td>$\lambda_{eff}$</td>
<td>The effective arrival rate</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Service rate</td>
</tr>
<tr>
<td>$\rho$</td>
<td>$\rho = \frac{\lambda}{\mu}$, offered load (offered traffic)</td>
</tr>
<tr>
<td>$p_k$</td>
<td>Stationary probabilities; $p_k$ is the probability that there are $k$ customers in the system</td>
</tr>
</tbody>
</table>

Some formulas for M/M/1 Queueing System

$N = N_q + N_s$
$T = W + \bar{x}$
$\bar{x} = \frac{1}{\mu}$
$p_k = p_0 \cdot \rho^k$
$p_0 = 1 - \rho$

$N = \frac{\rho}{1 - \rho}$
$T = \frac{1}{\mu - \lambda}$

$F_{\bar{s}}(t) = P(\bar{s} \leq t) = 1 - e^{-(\mu - \lambda)t}$

Little's formulas:

$N = \lambda_{eff} \cdot T$ (For an M/M/1 system $\lambda_{eff} = \lambda$, since there are no blocked customers)
$N_q = \lambda_{eff} \cdot W$
$N_s = \lambda_{eff} \cdot \bar{x}$
1. Consider an M/M/1 system in which customers arrive according to a Poisson process of rate $\lambda$. Service rate is $\mu = 20$ customers /minute.

The average number of customers is $N = 3$.

Calculate $\lambda$ and $W$.

Answer:
$\lambda = 15$,  
$W = 0.15$ minutes (=9 sec)

2. Consider an M/M/1 queueing system with arrival rate $\lambda$ and service rate $\mu$.

a) Derive the formula for the average number of customers in the system

$$N = \frac{\rho}{1 - \rho}$$

b) Calculate the average total time $T$ if the service rate is $\mu = 50$ customers /minute and the average number of customers is $N = 4$.

Solution:

a) M/M/1 system , rate transition diagram

From the rate transition diagram we have

$$p_k = p_0 \cdot \rho^k, \quad p_0 = 1 - \rho$$

The average number of the customers in the system can be calculated using the formula for the expectation of a discrete random variable:

$$N = \sum_{k=0}^{\infty} kp_k = \sum_{k=0}^{\infty} kp_0 \cdot \rho^k = p_0 \cdot \rho \sum_{k=0}^{\infty} kp^{k-1} \quad (*)$$

If we differentiate with respect to $\rho$ the formula for the sum of a geometric series

$$\sum_{k=0}^{\infty} \rho^k = \frac{1}{1-\rho}$$

we obtain

$$\sum_{k=0}^{\infty} kp^{k-1} = \frac{1}{(1-\rho)^2} \quad (**)$$

[Remark: $\frac{d}{d\rho} \left( \frac{1}{1-\rho} \right) = \frac{d}{d\rho} \left( (1-\rho)^{-1} \right) = (\text{chain rule}) = -(1-\rho)^{-2} \cdot (-1) = (1-\rho)^{-2} = \frac{1}{(1-\rho)^2}$]

We substitute (**) in (*) and get

$$N = p_0 \cdot \rho \sum_{k=0}^{\infty} kp^{k-1} = (1-\rho) \cdot \rho \cdot \frac{1}{(1-\rho)^2} = \frac{\rho}{1-\rho} \quad \text{which proves the formula.}$$
b) \[ N = \frac{\rho}{1 - \rho} \Rightarrow 4 = \frac{\rho}{1 - \rho} \Rightarrow \rho = \frac{4}{5} \]
Now \[ \rho = \frac{\lambda}{\mu} \Rightarrow \frac{4}{5} = \frac{\lambda}{50} \Rightarrow \lambda = 40. \]
From the Little’s formula
\[ N = \lambda_{\text{eff}} \cdot T, \] since \( \lambda_{\text{eff}} = \lambda \), we obtain
\[ 4 = 40 \cdot T \Rightarrow T = \frac{1}{10} \text{min} = 6 \text{sec} \]

3. An M/M/1 system has the service rate \( \mu = 10 \) customers per minute. Average time in the system for one customer is \( T = 3 \) minutes.

a) derive the formula \( T = \frac{1}{\mu - \lambda} \) (You can start with \( N = \frac{\rho}{1 - \rho} \))

b) Evaluate \( \lambda, N \) and \( \bar{x} \).

Answer: b) \( \lambda = 9.6666, N = 29, \bar{x} = 0.1 \) minutes.

4. A communication channel is operating at a transmission rate of 1000 000 bps.
To the channel arrive packets according to a Poisson process with rate \( \lambda = 100 \) packets per second. The packets have an exponentially distributed length with a mean of \( v = 5000 \) bits.
We assume that the channel can be modeled as an M/M/1 system with queueing discipline FCFS (First-Come-First-Served).
Calculate
a) \( \mu \) b) \( \rho \) c) \( \bar{x} \) c) \( N \) d) \( T \) e) \( W \) f) \( N_q \)

Answer:
\begin{align*}
\text{a) } & \mu = 200 \\
\text{b) } & \rho = 1/2 \\
\text{c) } & \bar{x} = 1/200 \text{ s} \\
\text{c) } & N = 1 \\
\text{d) } & T = 1/100 \text{ s} \\
\text{e) } & W = 1/200 \\
\text{f) } & N_q = 1/2 \\
\end{align*}

5. Data packets arrive to a communication node according to a Poisson process with an average rate of \( \lambda = 2400 \) packets per minute. The packets have exponentially distributed lengths with a mean of \( v = 1000 \) bits. A single outgoing communication link is operating at a transmission rate of \( K \) bits/second.
We assume that the link has a very large buffer so that it can be modeled as an M/M/1 system with queueing discipline FCFS (First-Come-First-Served).

a) Evaluate \( K \) if the average system time is \( T = 1 \) s.

For that value of \( K \) determine:

b) \( \mu \) c) \( N \) d) \( \bar{x} \) e) \( W \)

**Answer:**
\begin{align*}
\text{a) } & K = 41000 \text{ bits per second} \\
\text{b) } & \mu = 41 \text{ packets per second} \\
\text{c) } & N = 40 \\
\text{d) } & \bar{x} = \frac{1}{41} \text{ s} \\
\text{c) } & W = \frac{40}{41} \text{ s} \\
\end{align*}

6. Jobs (customers) arriving at an M/M/1 system according to a Poisson process with an average rate of 8 jobs per second. The Service rate is \( \mu = 10 \) jobs per second.
Find
a) The offered load
b) The probability that the system is idle (no customers in the system)
c) The probability that there are exact 2 customers in the system.
d) Average number of customers in the system
e) Average number of customers in the queue

**Answer:**
a) $\rho = 0.8$  
b) $p_0 = 0.2$.  
c) $p_2 = 0.128$  
d) $N = 4$  
e) $N_q = 3.2$

7. Consider an M/M/1 system in which customers arrive according to a Poisson process of rate $\lambda$. Service rate is $\mu = 10$ customers /second
The average system time is $T = 0.2$ s .
a) Calculate $\lambda$
b) Find $T$ if we replace server with a faster one, which has a service rate of $\mu = 40$ customers /second (but the arrival rate remains the same, $\lambda$ customers /second )

**Answer:**
a) $\lambda = 5$ customers /second,  
b) $T = 1/35$ s