M/G/1 Queueing System.

Service discipline: First come first served

M/G/1 Queueing System is a single-server queueing system with Poisson input, general service time distribution and unlimited number of waiting positions. Thus, an M/G/1 system has the following characteristics:

1. There is a single server with a general service time distribution with mean $\bar{x} = E(\bar{x})$ and second moment $\bar{x}^2 = E(\bar{x}^2)$.
   
   (The service rate is $\mu = \frac{1}{\bar{x}}$ customers per time unit)

2. Customers arriving according a Poisson process with the arrival rate $\lambda$ customers per time unit

3. Number of waiting positions = $\infty$

---

**Figure 1.**

------------------------------------------------------------------------
### Notation and Description

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N)</td>
<td>Average number of customers in the system, (N = N_q + N_s)</td>
</tr>
<tr>
<td>(N_q)</td>
<td>Average number of customers in the queue</td>
</tr>
<tr>
<td>(N_s)</td>
<td>Average number of customers in the service facilities</td>
</tr>
<tr>
<td>(\bar{x})</td>
<td>Random variable which describes time spent in the service facility by a customer</td>
</tr>
<tr>
<td>(\bar{x})</td>
<td>Average service time for a customer, (\bar{x} = E(\bar{x}))</td>
</tr>
<tr>
<td>(E(\bar{x}^2))</td>
<td>Second moment of the random variable (\bar{x})</td>
</tr>
<tr>
<td>(\sigma_x)</td>
<td>Standard deviation for (\bar{x}), (\sigma_x^2 = E(\bar{x}^2) - (E(\bar{x}))^2)</td>
</tr>
<tr>
<td>(C_{\bar{x}}^2)</td>
<td>Coefficient of variation of the service time, (C_{\bar{x}}^2 = \left(\frac{\sigma_x}{\bar{x}}\right)^2 = \frac{Var(\bar{x})}{(\bar{x})^2})</td>
</tr>
<tr>
<td>(\hat{w})</td>
<td>Random variable which describes time spent in the waiting queue by a customer</td>
</tr>
<tr>
<td>(W)</td>
<td>Average waiting time spent in the queue by a customer, (W = E(\hat{w}))</td>
</tr>
<tr>
<td>(\bar{s})</td>
<td>Random variable which describes time spent in the system by a customer; (\bar{s} = \bar{x} + \hat{w})</td>
</tr>
<tr>
<td>(T)</td>
<td>Average time spent in the system by a customer, (T = E(\bar{s})), (T = W + \bar{x})</td>
</tr>
<tr>
<td>(\lambda)</td>
<td>Arrival rate</td>
</tr>
<tr>
<td>(\lambda_{\text{eff}})</td>
<td>The effective arrival rate</td>
</tr>
<tr>
<td>(\mu)</td>
<td>Service rate</td>
</tr>
<tr>
<td>(\rho)</td>
<td>(\rho = \frac{\lambda}{\mu}), offered load (offered traffic)</td>
</tr>
<tr>
<td>(p_k)</td>
<td>Stationary probabilities; (p_k) is the probability that there are (k) customers in the system</td>
</tr>
</tbody>
</table>

### Some formulas for M/G/1 Queueing System

\[N = N_q + N_s, \quad T = W + \bar{x}, \quad \bar{x} = \frac{1}{\mu}\]

\[p_0 = 1 - \rho\]

### Little’s Formulas:

For an M/G/1 system \(\lambda_{\text{eff}} = \lambda\), since there are no blocked customers,

\[N_q = \lambda_{\text{eff}} \cdot W\]

\[N_s = \lambda_{\text{eff}} \cdot \bar{x}\]
**Pollaczeck - Khinchin formula:**

\[ W = \frac{\lambda \cdot E(\bar{x}^2)}{2(1 - \rho)} \quad \text{or} \quad W = \frac{\rho \cdot \bar{x}}{2(1 - \rho)}(1 + C_{\bar{x}}^2) \]

Total time in the system is then:

\[ T = \bar{x} + W = \bar{x} + \frac{\lambda \cdot E(\bar{x}^2)}{2(1 - \rho)} \]

Using Little’s theorem we find:

\[ N_q = \lambda W = \frac{\lambda^2 \cdot E(\bar{x}^2)}{2(1 - \rho)} \quad \text{or} \quad N_q = \frac{\rho^2}{2(1 - \rho)}(1 + C_{\bar{x}}^2) \]

\[ N_s = \lambda \bar{x} = \frac{\lambda}{\mu} = \rho \]

\[ N = N_s + N_q = \rho + \frac{\lambda^2 \cdot E(\bar{x}^2)}{2(1 - \rho)} \quad \text{or} \quad N = \rho + \frac{\rho^2}{2(1 - \rho)}(1 + C_{\bar{x}}^2) \]
### Some probability distributions

#### DISCRETE RANDOM VARIABLES

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Probability function $P(X = x)$</th>
<th>Mean $E[X]$</th>
<th>Variance $Var(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant $X = C$</td>
<td>1 if $X = C$ 0 if $X \neq C$</td>
<td>$C$</td>
<td>0</td>
</tr>
<tr>
<td>Bernoulli $p$ if $X = 1$ (1 - p) if $X = 0$</td>
<td>$p$</td>
<td>$p(1 - p)$</td>
<td></td>
</tr>
</tbody>
</table>
| Binomial Bin$(n,p)$   | \( \binom{n}{x} p^x (1-p)^{n-x} \)  
$x = 0, 1, \ldots, n$ | $np$ | $np(1-p)$ |
| Poisson $Po(\lambda)$| $e^{-\lambda} \frac{\lambda^x}{x!}$  
$x = 0, 1, 2, 3\ldots$ | $\lambda$ | $\lambda$ |
| Geometric, G(p)       | $p(1-p)^{x-1}$  
$x = 1, 2, 3\ldots$ | $\frac{1}{p}$ | $\frac{1 - p}{p^2}$ |
| Hypergeometric Hyp$(N,n,p)$ | \( \binom{Np}{x} \binom{N-n}{n-x} \) \( \binom{N}{n} \) | $np$ | $np(1-p)(N-n) / N - 1$ |

#### CONTINUOUS RANDOM VARIABLES

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Density function $f(x)$</th>
<th>Distribution function $F(x)$</th>
<th>Mean $E[X]$</th>
<th>Variance $Var(X)$</th>
</tr>
</thead>
</table>
| Uniform $U(a,b)$      | $\frac{1}{b-a}$, $a \leq x \leq b$  
$0$ otherwise | $\frac{x-a}{b-a}$, $a \leq x \leq b$ 
$0$ if $x < a$  
$1$ if $x > b$ | $\frac{a + b}{2}$ | $\frac{(b - a)^2}{12}$ |
| Exponential $Exp(\alpha)$ | $ae^{-\alpha x}$, $x \geq 0$ | $1 - e^{-\alpha x}$ | $\frac{1}{\alpha}$ | $\frac{1}{\alpha^2}$ |
| Normal $N(\mu, \sigma)$ | $\frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ | $\frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt$ | $\mu$ | $\sigma^2$ |
| Gamma $\Gamma(n, \alpha)$ | $\frac{\alpha^{n-1} e^{-\alpha x}}{\Gamma(n)}$, $x \geq 0$ | | $\frac{n}{\alpha}$ | $\frac{n}{\alpha^2}$ |
| Gamma=Erang dist. if $n$ is a positive integer) | | | | |
| Erlang $(n, \alpha)$   | $\frac{\alpha^{n-1} e^{-\alpha x}}{(n-1)!}$, $x \geq 0$ | | $\frac{n}{\alpha}$ | $\frac{n}{\alpha^2}$ |
**Q 1.** Find $\mu$, $\rho$, $E(\bar{x}^2)$, $C^2_{\bar{x}}$, $W$, $T$, $N_q$, $N_s$, and $N$ for an M/G/1 system if $\lambda = 1/40$ packets per millisecond, $\bar{x} = E(\bar{x}) = 10$ ms and the standard deviation of $\bar{x}$ is $\sigma = 1$ ms.

**Answer:**

$$\mu = 1/10, \quad \rho = 1/4, \quad E(\bar{x}^2) = 101, \quad C^2_{\bar{x}} = \frac{\sigma^2}{(E(\bar{x}))^2} = 1/100,$$

$W = 101/60$ ms, $T = 701/60$ ms

$N_q = 101/2400, \quad N_s = 1/4, \quad N = 701/2400$

**Q 2.** Find $\bar{x}$, $E(\bar{x}^2)$, $\mu$, $\rho$, $C^2_{\bar{x}}$, $W$, $T$, $N_q$, $N_s$ and $N$ for an M/G/1 system where $\lambda = 1/5$ packets per ms, and

$$f(t) = \begin{cases} \sin t & 0 \leq t \leq \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

is the density function of the random variable $\bar{x}$ (time unit = ms).

**Answer:**

$$\bar{x} = E(\bar{x}) = \int_0^{\pi/2} tf(t)dt = \int_0^{\pi/2} t \sin t dt = 1.$$

$$E(\bar{x}^2) = \int_0^{\pi/2} t^2 f(t)dt = \int_0^{\pi/2} t^2 \sin t dt = \pi - 2.$$

$$\mu = 1, \quad \rho = 1/5, \quad C^2_{\bar{x}} = \pi - 3, \quad W = \frac{\pi}{8} - \frac{1}{4} \approx 0.14 \text{ ms, } T = \frac{\pi}{8} + \frac{3}{4} \approx 1.14 \text{ ms}$$

$$N_q = \frac{\pi}{40} - \frac{1}{20} \approx 0.0285, \quad N_s = 1/5 = 0.2, \quad N = \frac{\pi}{40} + \frac{3}{20} \approx 0.2285$$

**Q 3.** Find and compare total system times $T$ for the following three M/G/1 queueing systems with the same arriving process of the rate $\lambda = 1/10$ packets per ms:

A) Mean service time is $\bar{x} = 5$ ms and service time $\bar{x}$ has an exponential distribution with parameter 1/5, $\bar{x} \in Exp(1/5)$.

B) Mean service time is $\bar{x} = 5$ ms and service time $\bar{x}$ is a random variable, uniformly distributed between 4ms and 6 ms, $\bar{x} \in Uniform[4, 6]$.

C) Service times are constant $\bar{x} = 5$ ms and therefore $\bar{x} = 5$ ms.

D) Mean service time is $\bar{x} = 5$ ms and service time $\bar{x}$ is an Erlang distributed random variable with $n=5$ and parameter 1, $\bar{x} \in Erlang(5, 1)$.

**Answer:**
A) Variance $= 25$, $E(\tilde{x}^2) = 50$, $T = 10$ ms
B) Variance $= 1/3$, $E(\tilde{x}^2) = 76/3$, $T \approx 7.53$ ms
C) Variance $= 0$, $E(\tilde{x}^2) = 25$, $T = 15/2 = 7.5$ ms
D) Variance $= 5$, $E(\tilde{x}^2) = 30$, $T = 8$ ms
So, for this case we have $T_c < T_b < T_d < T_a$.

Q 4. Compare waiting times $W$ for the
a) $M/M/1$, b) $M/U/1$ and c) $M/D/1$
queueing systems with arrival rate $\lambda$, average service time $\bar{x} = m$ and
a) exponential distributed service times $\tilde{x} \in \text{Exp}(\frac{1}{m})$,
b) uniform service times $\tilde{x} \in U(0,2m)$,
c) deterministic(constant) service times $\tilde{x} = \text{const} = m$.

Solution:
a) For an exp. random variable $\tilde{x} \in \text{Exp}(\frac{1}{m})$ we have $E(\tilde{x}) = m$ and $\text{Var}(\tilde{x}) = m^2$.
Thus
$$C_\tilde{x}^2 = \frac{\sigma^2}{(\tilde{x})^2} = \frac{\text{Var}(\tilde{x})}{(E(\tilde{x}))^2} = \frac{m^2}{m^2} = 1$$
and
$$W = \frac{\rho \cdot \tilde{x}}{2(1 - \rho)} (1 + C_\tilde{x}^2) = \frac{\rho \cdot m}{2(1 - \rho)} (1 + 1) = \frac{\rho \cdot m}{1 - \rho}$$

b) For an uniform random variable $\tilde{x} \in U(0,2m)$ $E(\tilde{x}) = m$ and $\text{Var}(\tilde{x}) = \frac{4m^2}{12} = \frac{m^2}{3}$.
Thus
$$C_\tilde{x}^2 = \frac{\sigma^2}{(\tilde{x})^2} = \frac{\text{Var}(\tilde{x})}{(E(\tilde{x}))^2} = \frac{m^2/3}{m^2} = 1/3$$
and
$$W = \frac{\rho \cdot \tilde{x}}{2(1 - \rho)} (1 + C_\tilde{x}^2) = \frac{\rho \cdot m}{2(1 - \rho)} (1 + 1/3) = \frac{2\rho \cdot m}{3(1 - \rho)}$$

c) For an constant random variable $\tilde{x} = \text{constant} = m$ we have $E(\tilde{x}) = m$ and $\text{Var}(\tilde{x}) = 0$.
Thus
$$C_\tilde{x}^2 = \frac{\sigma^2}{(\tilde{x})^2} = \frac{\text{Var}(\tilde{x})}{(E(\tilde{x}))^2} = \frac{0}{m^2} = 0$$
and
$$W = \frac{\rho \cdot \tilde{x}}{2(1 - \rho)} (1 + C_\tilde{x}^2) = \frac{\rho \cdot m}{2(1 - \rho)} (1 + 0) = \frac{\rho \cdot m}{2(1 - \rho)}$$
So, $W_{\text{const}} \leq W_{\text{unif}} \leq W_{\text{exp}}$ and the waiting time in an $M/D/1$ is half that in $M/M/1$. 
Q 5. Packets arrive at switching node according to a Poisson process with rate $\lambda = 2$ packets per millisecond. We regard the single outgoing link of the switching node as the only server. Packets service times (transit times) are uniformly distributed, $0 < \bar{x} \leq 0.2$ milliseconds. The input buffer has a very big capacity so that there are no blocked packets. We model this system as an M/G/1 queueing system.

Find: a) $\bar{x} = E(\bar{x})$, b) $\sigma^2 = Variance(\bar{x})$ c) $E(\bar{x}^2)$ d) $\mu$, $\rho$ e) $W$, $T$ f) $N_q$, $N_s$, $N$

Answer:
a) $\bar{x} = E(\bar{x}) = 0.1$, b) $\sigma^2 = Variance(\bar{x}) = 1/300$ c) $E(\bar{x}^2) = 1/75$ d) $\mu = 10$, $\rho = 1/5$

e) $W = 1/60$, $T = 7/60$ f) $N_q = 1/30$, $N_s = 1/5$, $N = 7/30$

Q 6. Packets arrive at switching node according to a Poisson process with rate $\lambda = 5$ packets per second. We regard the single outgoing link of the switching node as the only server, which has the capacity of $10^5$ bps. Packets length L are uniformly distributed, $0 < L \leq 10^4$ bits.

If we model this system as an M/G/1 queueing system find:

a) $\bar{x} = E(\bar{x})$, b) $\sigma^2 = Variance(\bar{x})$ c) $E(\bar{x}^2)$ d) $\mu$, $\rho$

e) $W$, $T$ f) $N_q$, $N_s$, $N$

Answer:

Since $\bar{x} = \frac{L}{v}$, we see that the packets service times are uniformly distributed, $0 < \bar{x} \leq 0.1$ seconds.

a) $\bar{x} = E(\bar{x}) = 1/20 = 0.05$ s, b) $\sigma^2 = Variance(\bar{x}) = \frac{0.1^2}{12} = 1/1200$ c) $E(\bar{x}^2) = 1/300$ d) $\mu = 20$, $\rho = 1/4$

e) $W = 1/90$, $T = 11/180$ f) $N_q = 1/18$, $N_s = 1/4$, $N = 11/36$

Q 7. Derive the Pollaczek - Khinchin formula:

$W = \frac{\lambda \cdot E(\bar{x}^2)}{2(1 - \rho)}$, or $W = \frac{\rho \cdot \bar{x}}{2(1 - \rho)} \left(1 + C_x^2\right)$

Solution:

Let $t_i$ denote interarrival times, that is time intervals between customers and let

$\bar{t} = E(t_i)$. Then $\bar{t} = \frac{1}{\lambda}$.

If the service discipline is first come, first served then the waiting time $\bar{w}$, spend by a customer arriving at time $t_i$ is the sum of the residual service time $R$ of the customer (if any) found in the server, and the $n_q$ service times of the customers (if any) found in queue.
So we have
\[ \tilde{W} = \sum_{i=1}^{n} x_i + R, \]
and
\[ E(\tilde{W}) = E(n_q) \cdot E(x_i) + E(R). \]
Hence
\[ W = N_q \cdot \bar{x} + E(R), \]
\[ W = \lambda W \cdot \bar{x} + E(R), \]
\[ W = \rho W + E(R), \]
Solving this equation for \( W \) we get
\[ W - \rho W = E(R), \]
and
\[ W = \frac{E(R)}{1 - \rho} \quad (*) \]
To find \( E(R) \), we sketch \( R(t) \) as a function of time \( t \). The residual service time is 0 when the system is empty.

Now \( E(R) = \lim_{t \to \infty} \frac{1}{t} \int_0^t R(t) dt \) (we calculate integral as the area under \( R(t) \))
\[
= \lim_{t \to \infty} \frac{1}{t} \left( \sum_{i=1}^{n} \frac{x_i^2}{2} \right) = \frac{1}{2} \left( \sum_{i=1}^{n} x_i^2 \right) \quad (\text{If we denote interarrival time intervals by } t_i \text{ then } t = \sum_{i=1}^{n} t_i) \]
\[
= \lim_{n \to \infty} \frac{1}{2} \left( \sum_{i=1}^{n} t_i \right) = \frac{1}{2} \\left( \sum_{i=1}^{n} t_i \right) / n = \frac{E(\tilde{x}^2)}{2E(t_i)} = \frac{E(\tilde{x}^2)}{2\bar{t}} = \frac{E(\tilde{x}^2)}{2(1/\lambda)} = \frac{\lambda E(\tilde{x}^2)}{2}\]
Thus \( E(R) = \frac{\lambda E(\bar{x}^2)}{2} \). (**)

Now, by substituting (**) in (*) we get

\[
W = \frac{\lambda E(\bar{x}^2)}{2(1 - \rho)}
\]

and that is what we wanted to prove.

Now it is not difficult to get another expression for the Pollaczek - Khinchin formula. Let \( \sigma_x \) denote the variance of a random variable \( X \). Then \( \sigma_x^2 = E(X^2) - (E(X))^2 \). By using this relation we obtain another expression for \( W \):

\[
W = \frac{\lambda E(\bar{x}^2)}{2(1 - \rho)} = \frac{\lambda (\bar{x}^2 + \sigma_x^2)}{2(1 - \rho)} = \frac{\lambda \cdot \bar{x}^2 (1 + \frac{\sigma_x^2}{\bar{x}^2})}{2(1 - \rho)} = \frac{\rho \cdot \bar{x} \cdot (1 + C_x^2)}{2(1 - \rho)},
\]

where \( C_x^2 = \frac{\sigma_x^2}{\bar{x}^2} \) is the coefficient of the variation of the service time.